Application of Wiener-Hermite Expansion to Strong Plasma Turbulence

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A Wiener-Hermite set of statistically orthogonal random functions in phase space is introduced for investigation of electrostatic plasma turbulence. Expansions of the random distribution functions in terms of the random base are considered and the equations governing the dynamics of the deterministic Wiener-Hermite Kernels are derived.

Introduction

Expansion of a random function in terms of an orthogonal random base was introduced by Cameron and Martin [1] and Wiener [2]. Meecham and Siegel [3] and Meecham and Jeng [4] applied this technique to the problem of hydrodynamic turbulence. Recently, Jahedi and Ahmadi [5] used it in their study of nonlinear structures subjected to random loads. The technique is now well known as the Wiener-Hermite expansion method.

The possible utility of the Wiener-Hermite expansion in closure of strong plasma turbulence was pointed out by Ahmadi [6].

In the present work the Wiener-Hermite method is applied to the problem of strong electrostatic plasma turbulence. Statistically orthogonal random base functions in phase space are introduced. The random distribution functions of ions and electrons are expanded in terms of the Wiener-Hermite set and the equations for the deterministic kernels are derived. "Closure" is achieved by discarding the forth and higher order terms in the Wiener-Hermite series. Deterministic evolution equations for the Wiener-Hermite kernel functions are derived and discussed.

Basic Equations

The equations governing the evolution of the distribution functions of electrons and ions in an electrostatic collisionless plasma are the well known Vlasov-Poisson equations. These are given by

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}\right) f_\alpha(\boldsymbol{X}, t) + \frac{e_\alpha}{m_\alpha} \sum_\mu e_\mu \int\int_{X^1} \mathrm{d}^3 X^1 \tag{1}$$

$$\times \frac{x_j - x_j^1}{|\boldsymbol{x} - \boldsymbol{x}^1|} \frac{\partial}{\partial v_j} [f_\mu(\boldsymbol{X}^1, t) f_\alpha(\boldsymbol{X}, t)] = 0, \quad \alpha = 1, 2,$$

where $\boldsymbol{X}$ stands for the pair $\boldsymbol{x}$, $\boldsymbol{v}$. Here, $\boldsymbol{x}$ is the position vector, $\boldsymbol{v}$ is the velocity vector, e_α and m_α are charges and masses of particles, $\alpha = 1, 2$ standing for electrons and ions, respectively. Summation convention is employed on Latin indices. When the plasma is in a turbulent state, the distribution functions become random functions of space and time.

To construct an appropriate Wiener-Hermite random base, we introduce a two component white noise process $\alpha_\alpha(\boldsymbol{X})$ defined in the phase space. $a_a(\boldsymbol{X})$ has the following statistical properties:

$$\langle a_\alpha(X) \rangle = 0, \quad \langle a_\alpha(\boldsymbol{X}^1)\, a_\beta(\boldsymbol{X}^2) \rangle = \delta_{\alpha\beta}\, \delta(\boldsymbol{X}^1 - \boldsymbol{X}^2), \tag{2}$$

where an angular bracket stands for the expected value. The elements of the Wiener-Hermite set are defined by

$$H^{(0)}(\boldsymbol{X}) = 1, \quad H_\alpha^{(1)}(\boldsymbol{X}) = a_\alpha(\boldsymbol{X}), \tag{3}$$

$$H_\alpha^{(2)}(\boldsymbol{X}^1, \boldsymbol{X}^2) = a_\alpha(\boldsymbol{X}^1)\, a_\beta(\boldsymbol{X}^2) - \delta_{\alpha\beta}\, \delta(\boldsymbol{X}^1 - \boldsymbol{X}^2), \ldots .$$

This set is a statistically orthogonal complete base. The expansions of the distribution functions in terms of the Wiener-Hermite set are given by

$$f_\alpha(\boldsymbol{X}) = F_\alpha^{(0)}(\boldsymbol{X}) + \sum_\beta \int\int_{X^1} F_{\alpha\beta}^{(1)}(\boldsymbol{X}, \boldsymbol{X}^1)\, H_\beta^{(1)}(\boldsymbol{X}^1)\, \mathrm{d}^6 X^1$$

$$+ \sum_\beta \sum_\gamma \int\int_{X^1} \int\int_{X^2} F_{\alpha\beta\gamma}^{(2)}(\boldsymbol{X}, \boldsymbol{X}^1, \boldsymbol{X}^2)$$

$$\times\, H_{\beta\gamma}^{(2)}(\boldsymbol{X}^1, \boldsymbol{X}^2)\, \mathrm{d}^6 X^1\, \mathrm{d}^6 X^2 + \ldots, \tag{4}$$

where $F_\alpha^{(0)}, F_{\alpha\beta}^{(1)}, \ldots$ are deterministic kernel functions whose time dependence is implicitly understood. The first term in the series is the mean value of the corresponding distribution function. The second term is the Gaussian part and the third and higher order terms correspond to the non-Gaussian part of the distribution functions. Substituting the series given by (4) into (1), multiplying by various elements of the Wiener-Hermite set taking expected values and making use of orthogonality properties,

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we find

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}\right) F_\alpha^{(0)}(\boldsymbol{X}) + \frac{e_\alpha}{m_\alpha} \sum_\mu e_\mu \int\int_{X^1} \mathrm{d}^6 X^1 \frac{x_j - x_j^1}{|\boldsymbol{x} - \boldsymbol{x}^1|^3} \times \frac{\partial}{\partial v_j} [F_\alpha^{(0)}(\boldsymbol{X}) F_\mu^{(0)}(\boldsymbol{X}^1) + \sum_\beta \int\int_{X^2} F_{\alpha\beta}^{(1)}(\boldsymbol{X}, \boldsymbol{X}^2) F_{\mu\beta}^{(1)}(\boldsymbol{X}^1, \boldsymbol{X}^2) \mathrm{d}^6 X^2] = 0, \tag{5}$$

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}\right) F_{\alpha\beta}^{(1)}(\boldsymbol{X}, \boldsymbol{X}') + \frac{e_\alpha}{m_\alpha} \sum_\mu e_\mu \int\int_{X^1} \mathrm{d}^6 X^1 \frac{x_j - x_j^1}{|\boldsymbol{x} - \boldsymbol{x}^1|^3} \times \frac{\partial}{\partial v_j} \{F_\mu^{(0)}(\boldsymbol{X}^1) F_{\alpha\beta}^{(1)}(\boldsymbol{X}, \boldsymbol{X}') + F_\alpha^{(0)}(\boldsymbol{X}) F_{\mu\beta}^{(1)}(\boldsymbol{X}^1, \boldsymbol{X}') + 2 \sum_\gamma \int\int_{X^2} \mathrm{d}^6 X^2 [F_{\alpha\gamma}^{(1)}(\boldsymbol{X}, \boldsymbol{X}^2) F_{\mu\beta\gamma}^{(2)}(\boldsymbol{X}^1, \boldsymbol{X}', \boldsymbol{X}^2) + F_{\mu\gamma}^{(1)}(\boldsymbol{X}^1, \boldsymbol{X}^2) F_{\alpha\beta\gamma}^{(2)}(\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{X}^2)]\} = 0, \tag{6}$$

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}\right) F_{\alpha\beta\gamma}^{(2)}(\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{X}'') + \frac{e_\alpha}{m_\mu} \sum_\mu e_\mu \int\int_{X^1} \mathrm{d}^6 X^1 \frac{x_j - x_j^1}{|\boldsymbol{x} - \boldsymbol{x}^1|^3} \frac{\partial}{\partial v_j} \times [\tfrac{1}{2} F_{\alpha\beta}^{(1)}(\boldsymbol{X}, \boldsymbol{X}') F_{\mu\gamma}^{(1)}(\boldsymbol{X}^1, \boldsymbol{X}'') + \tfrac{1}{2} F_{\alpha\gamma}^{(1)}(\boldsymbol{X}, \boldsymbol{X}'') F_{\mu\beta}^{(1)}(\boldsymbol{X}^1, \boldsymbol{X}') + F_\alpha^{(0)}(\boldsymbol{X}) F_{\mu\beta\gamma}^{(2)}(\boldsymbol{X}^1, \boldsymbol{X}', \boldsymbol{X}'') + F_\mu^{(0)}(\boldsymbol{X}^1) F_{\alpha\beta\gamma}^{(2)}(\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{X}'')] = 0. \tag{7}$$

Equations (6–7) are the basic equations governing the evolutions of the deterministic unknown kernels $F_\alpha^{(0)}$, $F_{\alpha\beta}^{(1)}$, and $F_{\alpha\beta\gamma}^{(2)}$. For given initial conditions, it is possible to find the time development of these functions. Equation (4) then provides explicit expressions for the distribution functions of ions and electrons. Various order statistics of the distribution functions could be generated by algebraic manipulation of the series expansion (4) and the known properties of the elements of the Wiener-Hermite set.

Further Remarks

Here, a formal derivation for the Wiener-Hermite series expansions of the distribution functions of ions and electrons of an electrostatic plasma in turbulent state is presented. The random base considered is time independent. However, generalizations to time dependent base could be carried out with out difficulty. Finally, it should be noted that the positive definitness of the distribution functions imposes further restrictions on the kernel functions which must be addressed before a numerical solution could be attempted.

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[1] R. H. Cameron and W. T. Martin, Ann. Math. **48,** 885 (1947).
[2] N. Wiener, Nonlinear Problem in Random Theory, John Wiley & Sons, Inc., New York 1958.
[3] W. C. Meecham and A. Siegel, Phys. Fluids **7,** 1178 (1969).
[4] W. C. Meecham and D. T. Jeng, J. Fluid Mech. **32,** 225 (1968).
[5] A. Jahedi and G. Ahmadi, J. Appl. Mech. Trans. ASME (in press).
[6] G. Ahmadi, Bull. Amer. Phys. Soc. **27,** 1203 (1983).